Int. J. Solids Structures Vol. 17, pp. 281–289
© Pergamon Press Ltd., 1981. Printed in Great Britain

0020-7683/81/0301-0281/$02.00

EFFECTS OF ROTARY INERTIA AND SHEAR ON NATURAL FREQUENCIES OF CONTINUOUS CIRCULAR CURVED BEAMS

TUNG-MING WANG
Department of Civil Engineering, University of New Hampshire, Durham, NH 03824, U.S.A.

and

MARC P. GUILBERT
General Dynamics/Electric Boat, Groton, CT 06340, U.S.A.

(*Received 2 January* 1980; *in revised form 5 June* 1980)

Abstract—The dynamic stiffness matrix for circular curved members of constant section has been derived for the determination of natural frequencies of continuous curved beams undergoing in-plane vibrations. An example of a two-span curved beam is given to illustrate the application of the proposed method and to show the effects of rotary inertia, shear and the central angle of the arc upon the natural frequencies of the beam.

1. INTRODUCTION

The problem of in-plane vibrations of curved beams has been the subject of many investigations. Den Hartog[1] in 1928 used the Rayleigh–Ritz method for finding the lowest natural frequency of circular arcs. The first detailed work dealing with the free vibration of pinned circular ring segments was published by Waltking[2]. Archer[3] carried out a mathematical study of the inextensional vibrations of an incomplete circular ring with additionsl terms to represent damping effects. Recently, Wang and Lee[4,5] presented a general method for analyzing both free and forced vibrations of multispan circular curved frames. Their method of analysis can also be used in the study of continuous curved beams.

The classical Bernoulli–Euler theory of flexural vibrations of beams has been recognized as adequate for relatively long slender beams at lower modes of vibration. For beams when the effect of the cross-sectional dimensions on frequencies cannot be neglected, and for beams in which higher modes are required, the Timoshenko theory[6] which considers the effects of rotary inertia and shear deformation gives a better approximation to the true behavior of a beam.

Much work has been done concerning the effects of rotary inertia and shear on straight beam vibrations. In case of curved beams, Philipson[7] studied the rotary inertia and shear effects on the in-plane vibrations of circular rings. The vibrations of a free ring subjected to the effects of bending, shear and extensional strain energies, together with translational and rotational kinetic energies were considered by Seidal and Erdelyi[8]. Rao and Sundararajan[9] investigated the in-plane flexural vibrations of free and stiffened rings with rotary inertia and shear effects being included.

In the works just mentioned, only the effects of rotary inertia and shear on single rings or ring segments have been considered. To the authors' knowledge, no investigations have been made for circular curved beams of multiple spans. The objective herein is to present a general method for analyzing continuous circular curved beams including both shear and rotary inertia effects. In this paper, the dynamic stiffness matrix for a circular curved member in terms of rotations, vertical and horizontal displacements, has been derived. The application of the proposed method is then illustrated by the determination of the natural frequencies of a two-span curved beam. Numerical results are given to show the effects of rotary inertia, shear deformation and the central angle of the arc upon the natural frequencies of the beam.

2. EQUATIONS OF MOTION AND THEIR SOLUTIONS

Consider the in-plane, small undamped vibration of a circular curved element ds as shown in Fig. 1.

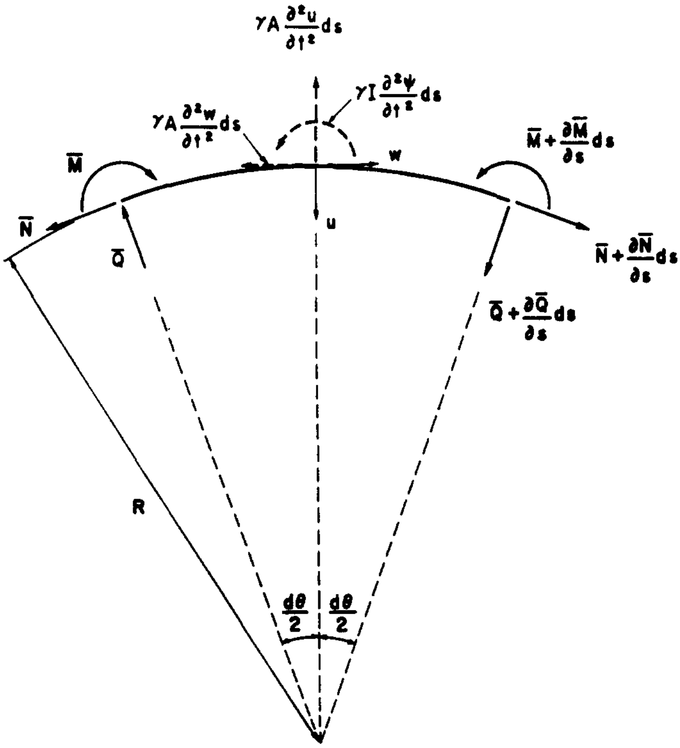

Fig. 1. Element of curved member subjected to forces and moments.

The equations of motion in radial and tangential directions and the moment equation are

$$\frac{\partial \bar{Q}}{\partial \theta}+\bar{N}=\gamma AR\frac{\partial^2 u}{\partial t^2} \tag{1}$$

$$\frac{\partial \bar{N}}{\partial \theta}-\bar{Q}=\gamma AR\frac{\partial^2 w}{\partial t^2} \tag{2}$$

$$-\frac{\partial \bar{M}}{\partial \theta}+\bar{Q}R=\gamma IR\frac{\partial^2 \psi}{\partial t^2} \tag{3}$$

where $\bar{Q}(\theta,t)$ is the shear force, $\bar{N}(\theta,t)$ the normal force, $\bar{M}(\theta,t)$ the bending moment, γ the mass per unit volume, I the moment of inertia of cross section, A the cross-sectional area, R the radius of circular arc, θ the angular coordinate, u the inward radial displacement, w the tangential displacement in the sense of increasing θ, ψ the bending slope and t the time. For inextensional vibration, the displacements must satisfy the following condition

$$u=\frac{\partial w}{\partial \theta}. \tag{4}$$

The total angle ϕ between the deformed and undeformed center lines may be expressed as[6]

$$\phi=\psi+\beta=\frac{1}{R}\left(w+\frac{\partial u}{\partial \theta}\right) \tag{5}$$

where β is the angular deformation due to shear.

From the elementary theory of bending, the bending moment and shear force are given respectively as follows:

$$\bar{M}=-\frac{EI}{R}\frac{\partial \psi}{\partial \theta} \tag{6}$$

$$\bar{Q} = kAG\beta \tag{7}$$

where E is the modulus of elasticity, G the modulus of rigidity and k the cross-sectional shape factor.

From eqns (4), (5) and (7) we obtain

$$\bar{Q} = \frac{kAG}{R}\left(\frac{\partial^2 w}{\partial \theta^2} + w - R\psi\right). \tag{8}$$

Eliminating $\bar{N}$ from eqns (1) and (2) and employing eqns (4) and (8) gives

$$R\psi + R\frac{\partial^2 \psi}{\partial \theta^2} = \frac{\partial^4 w}{\partial \theta^4} + 2\frac{\partial^2 w}{\partial \theta^2} + w - \frac{\gamma R^2}{kG}\frac{\partial^2 w}{\partial \theta^2 \partial t^2} + \frac{\gamma R^2}{kG}\frac{\partial^2 w}{\partial t^2}. \tag{9}$$

Substituting eqns (6) and (8) into eqn (3) yields

$$R\psi - \frac{EI}{kAGR}\frac{\partial^2 \psi}{\partial \theta^2} + \frac{\gamma IR}{kAG}\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 w}{\partial \theta^2} + w. \tag{10}$$

Finally, eliminating ψ from eqns (9) and (10), the following equation in w is obtained:

$$\frac{\partial^6 w}{\partial \theta^6} + 2\frac{\partial^4 w}{\partial \theta^4} + \frac{\partial^2 w}{\partial \theta^2} = \left(\frac{\gamma R^2}{E} + \frac{\gamma R^2}{kG}\right)\frac{\partial^6 w}{\partial \theta^4 \partial t^2} - \left(\frac{\gamma^2 R^4}{EkG}\right)\frac{\partial^6 w}{\partial \theta^2 \partial t^4}$$
$$+ \left(\frac{2\gamma R^2}{E} - \frac{\gamma R^2}{kG} - \frac{\gamma A R^4}{EI}\right)\frac{\partial^4 w}{\partial \theta^2 \partial t^2} + \left(\frac{\gamma^2 R^4}{EkG}\right)\frac{\partial^4 w}{\partial t^4} + \left(\frac{\gamma R^2}{E} + \frac{\gamma A R^4}{EI}\right)\frac{\partial^2 w}{\partial t^2}. \tag{11}$$

Assume that the curved member is undergoing free vibration with a frequency p and let

$$w(\theta, t) = W(\theta)\, e^{ipt} \tag{12}$$

$$\psi(\theta, t) = \Psi(\theta)\, e^{ipt}. \tag{13}$$

where $i = \sqrt{(-1)}$ and, $W(\theta)$ and $\Psi(\theta)$ are the normal functions of w and ψ, respectively. Substituting eqns (12) and (13) into eqns (9), (10) and (11) and omitting the common term e^{ipt}, one has

$$R\Psi + R\Psi'' = W^{IV} + (2 + b^2 s^2)W'' + (1 - b^2 s^2)W \tag{14}$$

$$(1 - b^2 r^2 s^2)R\Psi - s^2 R\Psi'' = W'' + W \tag{15}$$

$$W^{VI} + (2 + b^2 r^2 + b^2 s^2)W^{IV} + (1 - b^2 + 2b^2 r^2 - b^2 s^2 + b^4 r^2 s^2)W''$$
$$+ (b^2 + b^2 r^2 - b^4 r^2 s^2)W = 0 \tag{16}$$

where b, r, s represent effects of bending, rotary inertia and shear deformation, respectively, and are given by

$$b^2 = \gamma A p^2 R^4/(EI), \quad r^2 = I/(AR^2), \quad s^2 = EI/(kAGR^2) \tag{17}$$

and the primes for W and Ψ represent differentiation with respect to θ.

The solution of eqn (16) may be expressed as

$$W(\theta) = \sum_{n=1}^{6} c_n\, e^{\lambda_n \theta} \tag{18}$$

where c_n are constants to be determined by boundary conditions, and λ_n are the roots of the following polynomial equation

$$\lambda^6+(2+b^2r^2+b^2s^2)\lambda^4+(1-b^2+2b^2r^2-b^2s^2+b^4r^2s^2)\lambda^2 + (b^2+b^2r^2-b^4r^2s^2)=0. \tag{19}$$

For harmonic vibrations,

$$u(\theta,t)=U(\theta)\,\mathrm{e}^{ipt}. \tag{20}$$

From eqns (4) and (18) one has

$$U(\theta)=W'(\theta)=\sum_{n=1}^{6} c_n\lambda_n\,\mathrm{e}^{\lambda_n\theta}. \tag{21}$$

The relation between $\Psi(\theta)$ and $W(\theta)$ can be obtained from eqns (14) and (15) by eliminating Ψ''. Thus one obtains

$$(1+s^2-b^2r^2s^2)R\Psi=s^2W^{IV}+(1+2s^2+b^2s^4)W''+(1+s^2-b^2s^4)W \tag{22}$$

Substituting eqn (18) into eqn (22) yields

$$R\Psi(\theta)=\sum_{n=1}^{6} c_n v_n\,\mathrm{e}^{\lambda_n\theta} \tag{23}$$

where

$$v_n=\frac{s^2\lambda_n^4+(1+2s^2+b^2s^4)\lambda_n^2+(1+s^2-b^2s^4)}{(1+s^2-b^2r^2s^2)}. \tag{24}$$

3. DERIVATION OF DYNAMIC STIFFNESS MATRIX

Figure 2 shows a circular curved member of constant cross section subjected to harmonic displacements, linear and rotational, at the two ends A and B.

Let

$$\bar{M}(\theta,t)=M(\theta)\,\mathrm{e}^{ipt} \tag{25}$$

$$\bar{Q}(\theta,t)=Q(\theta)\,\mathrm{e}^{ipt} \tag{26}$$

$$\bar{N}(\theta,t)=N(\theta)\,\mathrm{e}^{ipt} \tag{27}$$

where M, Q and N are normal functions of $\bar{M}$, $\bar{Q}$ and $\bar{N}$, respectively.

Substituting eqns (12), (13) and (25)–(27) into eqns (6), (8) and (2) and omitting the common term e^{ipt} yield

$$M(\theta)=-\frac{EI}{R}\Psi'(\theta) \tag{28}$$

$$Q(\theta)=\frac{kAG}{R}\{W''(\theta)+W(\theta)-R\Psi(\theta)\} \tag{29}$$

$$N(\theta)=-Q'(\theta)-\gamma ARp^2W'(\theta). \tag{30}$$

Introducing eqns (18) and (23) into eqns (28)–(30) give

$$M(\theta)=-\frac{EI}{R^2}\sum_{n=1}^{6} c_n v_n\lambda_n\,\mathrm{e}^{\lambda_n\theta} \tag{31}$$

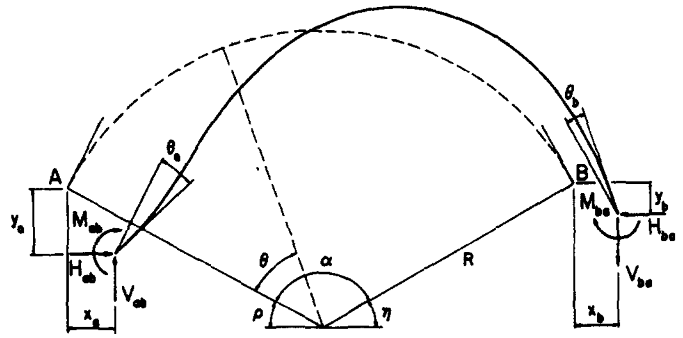

Fig. 2. Positive displacements, forces and moments with common factor e^{ipt} omitted.

$$Q(\theta) = \frac{EI}{R^3} \sum_{n=1}^{6} c_n z_n \, e^{\lambda_n \theta} \tag{32}$$

$$N(\theta) = -\frac{EI}{R^3} \sum_{n=1}^{6} c_n (z_n + b^2) \lambda_n \, e^{\lambda_n \theta} \tag{33}$$

where

$$z_n = \frac{b^2(s^2 - r^2) - \lambda_n^2(1 + b^2 r^2 + b^2 s^2) - \lambda_n^4}{(1 + s^2 - b^2 r^2 s^2)}. \tag{34}$$

Referring again to Fig. 2, the boundary conditions are

$$\left.\begin{aligned} \theta_a &= \Psi(0) \\ \theta_b &= \Psi(\alpha) \\ y_a &= U(0)\sin\rho - W(0)\cos\rho \\ y_b &= U(\alpha)\sin\eta + W(\alpha)\cos\eta \\ x_a &= U(0)\cos\rho + W(0)\sin\rho \\ x_b &= -U(\alpha)\cos\eta + W(\alpha)\sin\eta \end{aligned}\right\} . \tag{35}$$

Similarly, the moments, vertical and horizontal thrusts at the two ends may be expressed as

$$\left.\begin{aligned} M_{ab} &= M(0) \\ M_{ba} &= -M(\alpha) \\ V_{ab} &= Q(0)\sin\rho - N(0)\cos\rho \\ V_{ba} &= Q(\alpha)\sin\eta + N(\alpha)\cos\eta \\ H_{ab} &= -Q(0)\cos\rho - N(0)\sin\rho \\ H_{ba} &= Q(\alpha)\cos\eta - N(\alpha)\sin\eta \end{aligned}\right\} . \tag{36}$$

Substitution of eqns (18), (21), (23) and (31)–(33) into eqns (35) and (36) yield the results in the following matrix forms:

$$\mathbf{D} = \mathbf{AX} \tag{37}$$

$$\mathbf{F} = \frac{EI}{R^3}\mathbf{BX} \tag{38}$$

where

$$\mathbf{D} = \begin{bmatrix} \theta_a R \\ \theta_b R \\ y_a \\ y_b \\ x_a \\ x_b \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} M_{ab}/R \\ M_{ba}/R \\ V_{ab} \\ V_{ba} \\ H_{ab} \\ H_{ba} \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{bmatrix} \tag{39}$$

and matrices **A** and **B** are given in the Appendix.

Premultiplying eqn (37) by $\mathbf{A}^{-1}$ and substituting into eqn (38) one obtains

$$\mathbf{F} = \mathbf{SD} \tag{40}$$

where **S**, the dynamic stiffness matrix for a curved member, is given by

$$\mathbf{S} = \begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} \\ s_{21} & s_{22} & s_{23} & s_{24} & s_{25} & s_{26} \\ s_{31} & s_{32} & s_{33} & s_{34} & s_{35} & s_{36} \\ s_{41} & s_{42} & s_{43} & s_{44} & s_{45} & s_{46} \\ s_{51} & s_{52} & s_{53} & s_{54} & s_{55} & s_{56} \\ s_{61} & s_{62} & s_{63} & s_{64} & s_{65} & s_{66} \end{bmatrix} = \frac{EI}{R_3}\mathbf{BA}^{-1}. \tag{41}$$

4. EXAMPLE

A two-span symmetrical circular curved beam of constant section undergoing in-plane vertical vibrations as shown in Fig. 3 is analyzed for natural frequencies.

The boundary conditions are

$$\begin{aligned} &y_A = 0, \quad y_B = 0, \quad y_C = 0 \\ &x_A = 0, \quad x_B = 0, \quad x_C = 0 \\ &\theta_{BA} = \theta_{BC} \end{aligned} \tag{42}$$

and the equilibrium conditions are

$$\begin{aligned} &M_{AB} = 0, \quad M_{BA} + M_{BC} = 0, \quad M_{CB} = 0 \\ &V_{BA} = V_{BC} \\ &H_{BA} = H_{BC} \end{aligned} \tag{43}$$

Due to symmetry, Fig. 2 gives

$$\rho = \eta. \tag{44}$$

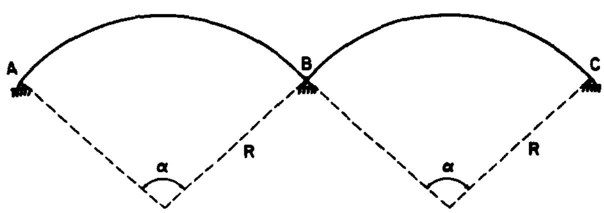

Fig. 3. A two-span circular curved beam.

Since the beam has two identical spans, one has

$$\mathbf{S}_{AB} = \mathbf{S}_{BC} = \mathbf{S}, \quad \mathbf{A}_{AB} = \mathbf{A}_{BC} = \mathbf{A}, \quad \mathbf{B}_{AB} = \mathbf{B}_{BC} = \mathbf{B}. \tag{45}$$

Thus eqns (40)–(42) give

$$\begin{aligned} M_{AB}/R &= s_{11}\theta_A R + s_{12}\theta_B R \\ M_{BA}/R &= s_{21}\theta_A R + s_{22}\theta_B R \\ M_{BC}/R &= s_{11}\theta_B R + s_{12}\theta_C R \\ M_{CB}/R &= s_{21}\theta_B R + s_{22}\theta_C R \end{aligned} \quad . \tag{46}$$

Substituting eqns (46) into eqns (43) yields a system of simultaneous equations in the following matrix form

$$\begin{bmatrix} M_{AB}/R \\ (M_{BA} + M_{BC})/R \\ M_{CB}/R \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} & 0 \\ s_{21} & s_{11} + s_{22} & s_{12} \\ 0 & s_{21} & s_{22} \end{bmatrix} \begin{bmatrix} \theta_A R \\ \theta_B R \\ \theta_C R \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{47}$$

Equating the determinant of the stiffness matrix in eqn (47) to zero yields the frequency equation as

$$\begin{vmatrix} s_{11} & s_{12} & 0 \\ s_{21} & s_{11} + s_{22} & s_{12} \\ 0 & s_{21} & s_{22} \end{vmatrix} = 0. \tag{48}$$

For a given curved beam with r and s known, the values of b_c ($c = 1, 2, 3, \ldots$) can be determined from eqn (48). In order to show the effects of rotary inertia and shear deformation on the natural frequencies of the beam, the beam section is assumed to be a rectangle. The elastic properties are $E = 206.85 \times 10^3$ M Pa (or 30×10^6 lb/in^2) and $G = 82.74 \times 10^3$ M Pa (or 12×10^6 lb/in^2). The value of k for a rectangular section as given by Timoshenko[10] is 0.667. Thus $E/(kG) \approx 4$ and $s \approx 2r$. Consider $\alpha = 60°$, the values of b for $r = 0$ and $r = 0.04$ for the first five modes, obtained from eqn (48), are respectively

$$b_0 = 33.63, \quad 42.94, \quad 75.08, \quad 86.97, \quad 141.6,$$

$$b = 29.82, \quad 35.45, \quad 59.45, \quad 63.51, \quad 99.03.$$

Let p_0 be the frequencies from the classical theory. Since $b/b_0 = p/p_0$, one has

$$p/p_0 = 0.887, \quad 0.826, \quad 0.792, \quad 0.730, \quad 0.699.$$

The results of p/p_0 vs r for $\alpha = 60°$, 120° and 180° for the first five modes, with r varying from 0 to 0.10, are shown in Fig. 4.

5. CONCLUSIONS

The dynamic stiffness matrix formulation for circular curved members of constant cross section, including the effects of rotary inertia and shear deformation, has been presented for the determination of the natural frequencies of continuous curved beams. The application of the proposed method has been illustrated in the example of a two-span curved beam undergoing natural vertical vibrations. From the curves shown in Fig. 4, it can be seen that the effects of rotary inertia and shear deformation become more pronounced as the central angle of the arc decreases. It is also observed that the reduction of the ratio of natural frequencies is increased as the values of r and s increase. For high modes the curves show that an increase in reduction as high as 63% is possible.

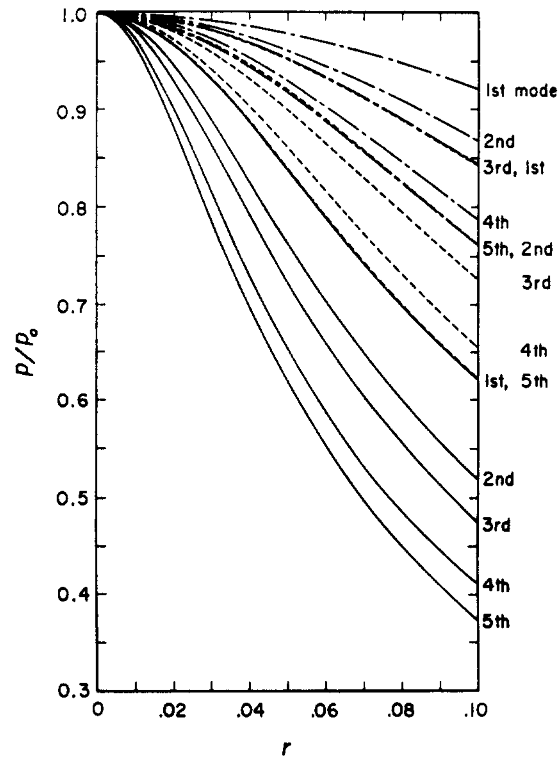

Fig. 4. Corrections in natural frequencies of a two-span curved beam owing to rotary inertia and shear deformation. ———, $\alpha = 60°$; - - - - -, $\alpha = 120°$; — — —, $\alpha = 180°$.

REFERENCES

1. J. P. Den Hartog, The lowest natural frequency of circular arcs. *Phil. Mag.* **5**, 400–408 (1928).
2. F. W. Waltking, Schwingungszahlen und schwingungsformen von kreisbogentragern. *Ing.-Arch.* **5**, 429–449 (1934).
3. R. R. Archer, Small vibrations of thin incomplete circular rings. *Int. J. Mech. Sci.* **1**, 45–56 (1960).
4. T. M. Wang and J. M. Lee, Natural frequencies of multi-span circular curved frames. *Int. J. Solids Structures* **8**, 791–805 (1972).
5. T. M. Wang and J. M. Lee, Forced vibrations of continuous circular arch frames. *J. Sound Vibration* **32**(2), 159–173 (1974).
6. S. P. Timoshenko, On the correction for shear of the differential equation for transverse vibrations of prismatic bars. *Phil. Mag.* **41**, 744–746 (1921).
7. L. L. Philipson, On the role of extension in the flexural vibrations of rings. *J. Appl. Mech.* **23**, 364–366 (1956).
8. B. S. Seidel and E. A. Erdely i, On the vibrations of a thick ring in its own plane. *J. Engng Industry* **86**, 240–244 (1964).
9. S. S. Rao and V. Sundararajan, In-plane flexural vibrations of circular rings. *J. Appl. Mech.* **36**, 620–625 (1969).
10. S. P. Timoshenko, *Strength of Materials*. Part I. D. Van Nostrand, New York (1955).

APPENDIX

Derivation of eqns (37) *and* (38)

Substituting eqns (18), (21), (23) into eqns (35) yield

$$\left.\begin{aligned}
\theta_a R &= \sum_{n=1}^{6} c_n v_n \\
\theta_b R &= \sum_{n=1}^{6} c_n v_n\, e^{\lambda_n \alpha} \\
y_a &= \sum_{n=1}^{6} c_n(\lambda_n \sin\rho - \cos\rho) \\
y_b &= \sum_{n=1}^{6} c_n(\lambda_n \sin\eta + \cos\eta)\, e^{\lambda_n\alpha} \\
x_a &= \sum_{n=1}^{6} c_n(\lambda_n \cos\rho + \sin\rho) \\
x_b &= \sum_{n=1}^{6} c_n(\sin\eta - \lambda_n \cos\eta)\, e^{\lambda_n\alpha}
\end{aligned}\right\} \quad (A1)$$

Equations (A1) may be written in matrix form as follows:

$$\mathbf{D} = \mathbf{A}\mathbf{X} \tag{37}$$

where $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ v_1 e^{\lambda_1\alpha} & v_2 e^{\lambda_2\alpha} & v_3 e^{\lambda_3\alpha} & v_4 e^{\lambda_4\alpha} & v_5 e^{\lambda_5\alpha} & v_6 e^{\lambda_6\alpha} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix} \tag{A2}$$

$a_{31} = \lambda_1 \sin\rho - \cos\rho$, $a_{32} = \lambda_2 \sin\rho - \cos\rho$, $a_{33} = \lambda_3 \sin\rho - \cos\rho$,
$a_{34} = \lambda_4 \sin\rho - \cos\rho$, $a_{35} = \lambda_5 \sin\rho - \cos\rho$, $a_{36} = \lambda_6 \sin\rho - \cos\rho$.
$a_{41} = (\lambda_1 \sin\eta + \cos\eta) e^{\lambda_1\alpha}$, $a_{42} = (\lambda_2 \sin\eta + \cos\eta) e^{\lambda_2\alpha}$,
$a_{43} = (\lambda_3 \sin\eta + \cos\eta) e^{\lambda_3\alpha}$, $a_{44} = (\lambda_4 \sin\eta + \cos\eta) e^{\lambda_4\alpha}$,
$a_{45} = (\lambda_5 \sin\eta + \cos\eta) e^{\lambda_5\alpha}$, $a_{46} = (\lambda_6 \sin\eta + \cos\eta) e^{\lambda_6\alpha}$.
$a_{51} = \lambda_1 \cos\rho + \sin\rho$, $a_{52} = \lambda_2 \cos\rho + \sin\rho$, $a_{53} = \lambda_3 \cos\rho + \sin\rho$,
$a_{54} = \lambda_4 \cos\rho + \sin\rho$, $a_{55} = \lambda_5 \cos\rho + \sin\rho$, $a_{56} = \lambda_6 \cos\rho + \sin\rho$.
$a_{61} = (\sin\eta - \lambda_1 \cos\eta) e^{\lambda_1\alpha}$, $a_{62} = (\sin\eta - \lambda_2 \cos\eta) e^{\lambda_2\alpha}$,
$a_{63} = (\sin\eta - \lambda_3 \cos\eta) e^{\lambda_3\alpha}$, $a_{64} = (\sin\eta - \lambda_4 \cos\eta) e^{\lambda_4\alpha}$,
$a_{65} = (\sin\eta - \lambda_5 \cos\eta) e^{\lambda_5\alpha}$, $a_{66} = (\sin\eta - \lambda_6 \cos\eta) e^{\lambda_6\alpha}$.

Similarly, substituting eqns (31)–(33) into eqns (36) give

$$\left.\begin{aligned} M_{ab}/R &= -(EI/R^3) \sum_{n=1}^{6} c_n v_n \lambda_n \\ M_{ba}/R &= (EI/R^3) \sum_{n=1}^{6} c_n v_n \lambda_n e^{\lambda_n\alpha} \\ V_{ab} &= (EI/R^3) \sum_{n=1}^{6} c_n \{z_n(\sin\rho + \lambda_n \cos\rho) + b^2\lambda_n \cos\rho\} \\ V_{ba} &= (EI/R^3) \sum_{n=1}^{6} c_n \{z_n(\sin\eta - \lambda_n \cos\eta) - b^2\lambda_n \cos\eta\} e^{\lambda_n\alpha} \\ H_{ab} &= (EI/R^3) \sum_{n=1}^{6} c_n \{z_n(\lambda_n \sin\rho - \cos\rho) + b^2\lambda_n \sin\rho\} \\ H_{ba} &= (EI/R^3) \sum_{n=1}^{6} c_n \{z_n(\lambda_n \sin\eta + \cos\eta) + b^2\lambda_n \sin\eta\} e^{\lambda_n\alpha} \end{aligned}\right\}. \tag{A3}$$

In matrix form, eqns (A3) may be expressed as

$$\mathbf{F} = \frac{EI}{R^3}\mathbf{B}\mathbf{X} \tag{38}$$

where $\mathbf{B}$ is given by

$$\mathbf{B} = \begin{bmatrix} -v_1\lambda_1 & -v_2\lambda_2 & -v_3\lambda_3 & -v_4\lambda_4 & -v_5\lambda_5 & -v_6\lambda_6 \\ v_1\lambda_1 e^{\lambda_1\alpha} & v_2\lambda_2 e^{\lambda_2\alpha} & v_3\lambda_3 e^{\lambda_3\alpha} & v_4\lambda_4 e^{\lambda_4\alpha} & v_5\lambda_5 e^{\lambda_5\alpha} & v_6\lambda_6 e^{\lambda_6\alpha} \\ b_{31} & b_{32} & b_{33} & b_{34} & b_{35} & b_{36} \\ b_{41} & b_{42} & b_{43} & b_{44} & b_{45} & b_{46} \\ b_{51} & b_{52} & b_{53} & b_{54} & b_{55} & b_{56} \\ b_{61} & b_{62} & b_{63} & b_{64} & b_{65} & b_{66} \end{bmatrix} \tag{A4}$$

$b_{31} = z_1(\sin\rho + \lambda_1 \cos\rho) + b^2\lambda_1 \cos\rho$, $b_{32} = z_2(\sin\rho + \lambda_2 \cos\rho) + b^2\lambda_2 \cos\rho$,
$b_{33} = z_3(\sin\rho + \lambda_3 \cos\rho) + b^2\lambda_3 \cos\rho$, $b_{34} = z_4(\sin\rho + \lambda_4 \cos\rho) + b^2\lambda_4 \cos\rho$,
$b_{35} = z_5(\sin\rho + \lambda_5 \cos\rho) + b^2\lambda_5 \cos\rho$, $b_{36} = z_6(\sin\rho + \lambda_6 \cos\rho) + b^2\lambda_6 \cos\rho$.
$b_{41} = \{z_1(\sin\eta - \lambda_1 \cos\eta) - b^2\lambda_1 \cos\eta\} e^{\lambda_1\alpha}$, $b_{42} = \{z_2(\sin\eta - \lambda_2 \cos\eta) - b^2\lambda_2 \cos\eta\} e^{\lambda_2\alpha}$,
$b_{43} = \{z_3(\sin\eta - \lambda_3 \cos\eta) - b^2\lambda_3 \cos\eta\} e^{\lambda_3\alpha}$, $b_{44} = \{z_4(\sin\eta - \lambda_4 \cos\eta) - b^2\lambda_4 \cos\eta\} e^{\lambda_4\alpha}$,
$b_{45} = \{z_5(\sin\eta - \lambda_5 \cos\eta) - b^2\lambda_5 \cos\eta\} e^{\lambda_5\alpha}$, $b_{46} = \{z_6(\sin\eta - \lambda_6 \cos\eta) - b^2\lambda_6 \cos\eta\} e^{\lambda_6\alpha}$.
$b_{51} = z_1(\lambda_1 \sin\rho - \cos\rho) + b^2\lambda_1 \sin\rho$, $b_{52} = z_2(\lambda_2 \sin\rho - \cos\rho) + b^2\lambda_2 \sin\rho$,
$b_{53} = z_3(\lambda_3 \sin\rho - \cos\rho) + b^2\lambda_3 \sin\rho$, $b_{54} = z_4(\lambda_4 \sin\rho - \cos\rho) + b^2\lambda_4 \sin\rho$,
$b_{55} = z_5(\lambda_5 \sin\rho - \cos\rho) + b^2\lambda_5 \sin\rho$, $b_{56} = z_6(\lambda_6 \sin\rho - \cos\rho) + b^2\lambda_6 \sin\rho$.
$b_{61} = \{z_1(\lambda_1 \sin\eta + \cos\eta) + b^2\lambda_1 \sin\eta\} e^{\lambda_1\alpha}$, $b_{62} = \{z_2(\lambda_2 \sin\eta + \cos\eta) + b^2\lambda_2 \sin\eta\} e^{\lambda_2\alpha}$,
$b_{63} = \{z_3(\lambda_3 \sin\eta + \cos\eta) + b^2\lambda_3 \sin\eta\} e^{\lambda_3\alpha}$, $b_{64} = \{z_4(\lambda_4 \sin\eta + \cos\eta) + b^2\lambda_4 \sin\eta\} e^{\lambda_4\alpha}$,
$b_{65} = \{z_5(\lambda_5 \sin\eta + \cos\eta) + b^2\lambda_5 \sin\eta\} e^{\lambda_5\alpha}$, $b_{66} = \{z_6(\lambda_6 \sin\eta + \cos\eta) + b^2\lambda_6 \sin\eta\} e^{\lambda_6\alpha}$.